IV. "On the Dynamical Theory of the Tides of Long Period." By G. H. Darwin, LL.D., F.R.S., Fellow of Trinity College and Plumian Professor in the University of Cambridge. Received November 5, 1886.

In the following note an objection is raised against Laplace's method of treating these tides, and a dynamical solution of the problem, founded on a paper by Sir William Thomson, is offered.

Let θ, ϕ be the colatitude and longitude of a point in the ocean, let ξ and $\eta \sin\theta$ be the displacements from its mean position of the water occupying that point at the time t, let $\mathfrak{h}$ be the height of the tide, and let $\mathfrak{e}$ be the height of the tide according to the equilibrium theory; let n be the angular velocity of the earth's rotation, g gravity, a the earth's radius, and γ the depth of the ocean at the point θ, ϕ.

Then Laplace's equations of motion for tidal oscillations are—

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2}-2n\sin\theta\cos\theta\frac{d\eta}{dt} &= -\frac{g}{a}\frac{d}{d\theta}(\mathfrak{h}-\mathfrak{e}) \\ \sin\theta\frac{d^2\eta}{dt^2}+2n\cos\theta\frac{d\xi}{dt} &= -\frac{g}{a\sin\theta}\frac{d}{d\phi}(\mathfrak{h}-\mathfrak{e}) \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

And the equation of continuity is—

$$\mathfrak{h}a+\frac{1}{\sin\theta}\frac{d}{d\theta}(\gamma\xi\sin\theta)+\gamma\frac{d\eta}{d\phi}=0. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The only case which will be considered here is where the depth of the ocean is constant, and we shall only treat the oscillations of long period in which the displacements are not functions of the longitude.

As the motion to be considered only involves steady oscillation, we assume—

$$\left.\begin{aligned} \mathfrak{e} &= e\cos(2nft+\alpha) \\ \mathfrak{h} &= h\cos(2nft+\alpha) \\ \xi &= x\cos(2nft+\alpha) \\ \eta &= y\sin(2nft+\alpha) \\ u &= h-e \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence, by substitution in (1), we have

$$\left.\begin{aligned} xf^2+yf\sin\theta\cos\theta &= \frac{1}{4m}\frac{du}{d\theta} \\ yf^2\sin\theta+xf\cos\theta &= 0 \end{aligned}\right\},$$

where $$m=\frac{n^2a}{g}.$$

Whence $$x(f^2-\cos^2\theta)=\frac{1}{4m}\frac{du}{d\theta},$$

$$y\sin\theta(f^2-\cos^2\theta)=-\frac{1}{4m}\frac{\cos\theta}{f}\frac{du}{d\theta}.$$

Then substituting for x and y in (2), which, when γ is constant and η is not a function of ϕ, becomes

$$ha+\frac{\gamma}{\sin\theta}\frac{d}{d\theta}(\xi\sin\theta)=0,$$

we get $$\frac{\gamma}{\sin\theta}\frac{d}{d\theta}\left[\frac{\sin\theta\, du/d\theta}{f^2-\cos^2\theta}\right]+4ma(u+e)=0.$$

This is Laplace's equation for tidal oscillations of the first kind.* In these tides f is a small fraction, being about $\frac{1}{28}$ in the case of the fortnightly tide, and e the coefficient in the equilibrium tide is equal to $E(\frac{1}{3}-\cos^2\theta)$, where E is a known function of the elements of the orbit of the tide-generating body, and of the obliquity of the ecliptic.

If now we write $\beta=4ma/\gamma$, and $\mu=\cos\theta$, our equation becomes

$$\frac{d}{d\mu}\left[\frac{1-\mu^2}{\mu^2-f^2}\frac{du}{d\mu}\right]=\beta[u+E(\tfrac{1}{3}-\mu^2)]. \quad . \quad . \quad . \quad . \quad (4)$$

In treating these oscillations Laplace does not use this equation, but seeks to show that friction suffices to make the ocean assume at each instant its form of equilibrium. His conclusion is no doubt true, but the question remains as to what amount of friction is to be regarded as sufficing to produce the result, and whether oceanic tidal friction can be great enough to have the effect which he supposes it to have.

The friction here contemplated is such that the integral effect is represented by a retarding force proportional to the velocity of the fluid relatively to the bottom. Although proportionality to the square of the velocity would probably be nearer to the truth, yet Laplace's hypothesis suffices for the present discussion. In oscillations of the class under consideration, the water moves for half a period north, and then for half a period south.

Now in systems where the resistances are proportional to velocity, it is usual to specify the resistance by a modulus of decay, namely, that period in which a velocity is reduced by friction to e^{-1} or $1\div2{\cdot}783$ of its initial value; and the friction contemplated by Laplace

* 'Mécanique Céleste.'

is such that the modulus of decay is short compared with the semi-period of oscillation.

The quickest of the tides of long period is the fortnightly tide, hence for the applicability of Laplace's conclusion, the modulus of decay must be short compared with a week. Now it seems practically certain that the friction of the ocean bed would not much affect the velocity of a slow ocean current in a day or two. Hence we cannot accept Laplace's hypothesis as to the effect of friction.

We now, therefore, proceed to the solution of the equation of motion when friction is entirely neglected.

The solution here offered is indicated in a footnote to a paper by Sir William Thomson ('Phil. Mag.,' vol. 50, 1875, p. 280), but has never been worked out before.

The symmetry of the motion demands that u, when expanded in a series of powers of μ, shall only contain even powers of μ.

Let us assume then

$$\frac{1}{\mu^2-f^2}\frac{du}{d\mu}=B_1\mu+B_3\mu^3+\ldots+B_{2i+1}\mu^{2i+1}+\ldots$$

Then

$$\frac{1-\mu^2}{\mu^2-f^2}\frac{du}{d\mu}=B_1\mu+(B_3-B_1)\mu^3+\ldots+(B_{2i+1}-B_{2i-1})\mu^{2i+1}+\ldots$$

$$\frac{d}{d\mu}\left[\frac{1-\mu^2}{\mu^2-f^2}\frac{du}{d\mu}\right]=B_1+3(B_3-B_1)\mu^2+\ldots+(2i+1)(B_{2i+1}-B_{2i-1})\mu^{2i}+\ldots \quad (5)$$

Again

$$\frac{du}{d\mu}=-f^2B_1\mu+(B_1-f^2B_3)\mu^3+\ldots+(B_{2i-1}-f^2B_{2i+1})\mu^{2i+1}+\ldots$$

$$u=C-\tfrac{1}{2}f^2B_1{}^2+\tfrac{1}{4}(B_1-f^2B_3)\mu^4+\ldots+\tfrac{1}{2i}(B_{2i-3}-f^2B_{2i-1})\mu^{2i}+\ldots, \quad (6)$$

where C is a constant.

Then substituting from (5) and (6) in (4), and equating to zero the successive coefficients of the powers of μ, we find,

$$\left.\begin{array}{l}
C=-\tfrac{1}{3}E+B_1/\beta \\
B_3-B_1(1-\tfrac{1}{2.3}f^2\beta)+\tfrac{1}{3}\beta E=0 \\
\cdots\cdots\cdots\cdots \\
B_{2i+1}-B_{2i-1}(1-\tfrac{1}{2i(2i+1)}f^2\beta)-\tfrac{1}{2i(2i+1)}\beta B_{2i-3}=0 \\
\cdots\cdots\cdots\cdots
\end{array}\right\} \quad (7)$$

Thus the constants C and B_3, B_5, &c., are all expressible in terms of B_1.

We may remark that if

$$-\tfrac{1}{2}\cdot\tfrac{1}{3}\cdot\beta B_{-1}=\tfrac{1}{3}\beta E, \text{ or } B_{-1}=-2E,$$

then the general equation of condition in (7) may be held to apply for all values of i from 1 to infinity.

Let us now write it in the form—

$$\frac{B_{2i+1}}{B_{2i-1}}=1-\frac{1}{2i(2i+1)}f^2\beta+\frac{1}{2i(2i+1)}\beta\frac{B_{2i-3}}{B_{2i-1}}. \quad . \quad . \quad . \quad . \quad (8)$$

When i is large, B_{2i+1}/B_{2i-1} either tends to become infinitely small, or it does not do so.

Let us suppose that it does not tend to become infinitely small. Then it is obvious that the successive B's tend to become equal to one another, and so also do the coefficients $B_{2i-1}-f^2B_{2i+1}$ in the expression for $du/d\mu$.

Hence $\dfrac{du}{d\mu}=L+\dfrac{M}{1-\mu^2}$, where L, M are finite, for all values of μ.

Hence $\dfrac{du}{d\theta}=-L\sqrt{1-\mu^2}+\dfrac{M}{\sqrt{1-\mu^2}}$, and therefore x is infinite when $\mu=1$ at the pole, and $d\xi/dt$ is infinite there also.

Hence the hypothesis, that B_{2i+1}/B_{2i-1} does not tend to become infinitely small, gives us infinite velocity at the pole. But with a globe covered with water this is impossible, the hypothesis is negatived, and B_{2i+1}/B_{2i-1} tends to become infinitely small.

This being established let us write (8) in the form—

$$\frac{B_{2i-1}}{B_{2i-3}}=\frac{-\frac{1}{2i(2i+1)}\beta}{1-\frac{1}{2i(2i+1)}f^2\beta-B_{2i+1}/B_{2i-1}}. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

By repeated applications of (9) we have in the form of a continued fraction

$$\frac{B_{2i-1}}{B_{2i-3}}=\frac{-\frac{1}{2i(2i+1)}\beta}{1-\frac{1}{2i(2i+1)}f^2\beta}\,\overline{\Big|+}\,\frac{\frac{1}{(2i+2)(2i+3)}\beta}{1-\frac{1}{(2i+2)(2i+3)}f^2\beta}\,\overline{\Big|+}\,\frac{\frac{1}{(2i+4)(2i+5)}\beta}{1-\frac{1}{(2i+4)(2i+5)}f^2\beta}\,\overline{\Big|+}\,\&c.$$

$$. \quad . \quad . \quad (10)$$

And we know that this is a continuous approximation, which must hold in order to satisfy the condition that the water covers the whole globe.

Let us denote this continued fraction by $-N_i$.

Then, if we remember that $B_{-1}=-2E$, we have

$$B_1=2EN_1, \qquad \frac{B_3}{B_1}=-N_2, \qquad \frac{B_5}{B_3}=+N_3, \text{ \&c.},$$

so that

$$B_3=-2EN_1N_2, \qquad B_5=-2EN_1N_2N_3, \qquad B_7=-2EN_1N_2N_3N_4, \text{ \&c.}$$

and
$$C=-\tfrac{1}{3}E+2\frac{EN_1}{\beta}.$$

Then the height of tide $\mathfrak{h}$ is equal to $h\cos(2nft+\alpha)$, the equilibrium tide $\mathfrak{e}$ is equal to $E(\frac{1}{3}-\mu^2)\cos(2nft+\alpha)$, and we have

$$h=u+E(\tfrac{1}{3}-\mu^2)$$
$$=C+\tfrac{1}{3}E-(E+\tfrac{1}{2}f^2B_1)\mu^2+\tfrac{1}{4}(B_1-f^2B_3)\mu^4+\tfrac{1}{6}(B_3-f^2B_5)\mu_6+\dots$$
$$\frac{h}{E}=\frac{2N_1}{\beta}-(1+f^2N_1)\mu^2+\tfrac{1}{2}N_1(1+f^2N_2)\mu^4-\tfrac{1}{3}N_1N_2(1+f^2N_3)\mu_6+\dots$$

Now when $\beta=40$, we have $\gamma=\frac{1}{40}\times 4ma=\frac{1}{2900}a=7260$ feet; so that $\beta=40$ gives an ocean of 1200 fathoms.

With this value of β, and with $f=\cdot 0365012$, which is the value for the fortnightly tide, I find

$$N_1=3\cdot 040692,\quad N_2=1\cdot 20137,\quad N_3=\cdot 66744,\quad N_4=\cdot 42819,\quad N_5=\cdot 29819,$$
$$N_6=\cdot 21950,\quad N_7=\cdot 16814,\quad N_8=\cdot 13287,\quad N_9=\cdot 107,\quad N_{10}=\cdot 1, \text{ \&c.}$$

These values give

$$\frac{2}{\beta}N_1=\cdot 15203, \qquad 1+f^2N_1=1\cdot 0041, \qquad \tfrac{1}{2}N_1(1+f^2N_2)=1\cdot 5228,$$
$$\tfrac{1}{3}N_1N_2(1+f^2N_3)=1\cdot 2187, \qquad \tfrac{1}{4}N_1N_2N_3(1+f^2N_4)=\cdot 6099,$$
$$\tfrac{1}{5}N_1\dots N_4(1+f^2N_5)=\cdot 2089 \qquad \tfrac{1}{6}N_1\dots N_5(1+f^2N_6)=\cdot 0519,$$
$$\tfrac{1}{7}N_1\dots N_6(1+f^2N_7)=\cdot 0098, \qquad \tfrac{1}{8}N_1\dots N_7(1+f^2N_8)=\cdot 0014,$$
$$\tfrac{1}{9}N_1\dots N_8(1+f^2N_9)=\cdot 00017, \text{ \&c.}$$

So that

$$\frac{h}{E}=\cdot 1520-1\cdot 0041\mu^2+1\cdot 5228\mu^4-1\cdot 2187\mu^6+\cdot 6099\mu^8-\cdot 2089\mu^{10}$$
$$+\cdot 0519\mu^{12}-\cdot 0098\mu^{14}+\cdot 0014\mu^{16}-\cdot 0002\mu^{18}+\dots$$

At the pole, where $\mu=1$, the equilibrium tide is $-\frac{2}{3}E$; at the equator it is $+\frac{1}{3}E$.

Now at the pole $h=-E\times\cdot 1037=-\frac{2}{3}E\times\cdot 1556$,

and at the equator $h=+E\times\cdot 1520=\ \frac{1}{3}E\times\cdot 4561$.

In a second case, namely, with an ocean four times as deep, so that $\beta=10$, I find

$$\frac{h}{E} = \cdot 2363 - 1\cdot 0016\mu^2 + \cdot 5910\mu^4 - \cdot 1627\mu^6 + \cdot 0258\mu^8 - \cdot 0026\mu^{10} + \cdot 0002\mu^{12} - \ldots$$

At the pole $\qquad h = -E \times \cdot 3137 = -\frac{2}{3}E \times \cdot 471,$

at the equator $\qquad h = +E \times \cdot 2363 = +\frac{1}{3}E \times \cdot 709.$

With a deeper ocean we should soon arrive at the equilibrium value for the tide, for N_2, N_3, &c., become very small, and $2N_1/\beta$ becomes equal to $\frac{1}{3}$.

These two cases, $\beta = 40$, $\beta = 10$, are two of those for which Laplace has given solutions in the case of the semi-diurnal and diurnal tides. We notice that, with such oceans as we have to deal with, the tide of long period is certainly less than half its equilibrium amount.

In Thomson and Tait's 'Natural Philosophy' (edition of 1883) I have made a comparison of the observed tides of long period with the equilibrium theory. The probable errors of the results are large, but not such as to render them worthless, and in view of the present investigation it is surprising to find that on the average the tides of long period amount to as much as two-thirds of their equilibrium value.

The investigation in the 'Natural Philosophy' was undertaken in the belief of the correctness of Laplace's view as to the tides of long period, and was intended to evaluate the effective rigidity of the earth's mass.

The present result shows us that it is not possible to attain any estimate of the earth's rigidity in this way, but as the tides of long period are distinctly sensible, we may accept the investigation in the 'Natural Philosophy' as generally confirmatory of Thomson's view as to the great effective rigidity of the whole earth's mass.

There is one tide, however, of long period of which Laplace's argument from friction must hold true. In consequence of the regression of the nodes of the moon's orbit there is a minute tide with a period of nearly nineteen years, and in this case friction must be far more important than inertia. Unfortunately this tide is very minute, and as I have shown in a Report for 1886 to the British Association on the tides, it is entirely masked by oscillations of sea level produced by meteorological or other causes.

Thus it does not seem likely that it will ever be possible to evaluate the effective rigidity of the earth's mass by means of tidal observations.